Probabilistic Methods in Option Pricing

Overview

(Very brief) Introduction to financial mathematics and the link with probability theory.

Focus on (equity) derivative pricing.

Introduction of the Parisian option contract.

Pricing the Parisian contract with Laplace transforms

A side result: an expression for the probability of an event related to excursions of Brownian motion.
The most extensively traded options plain vanilla call and put contracts.

Pay-off Call
Pay-off Put

X, Strike
$S_T$
How is it possible that there is a quotation for every expiry and strike?

Is Royal Dutch moving up or down this day?

Suppose we can model the behavior of a stock as follows,

And we are interested in a call 100 option

What is the relation between the call option prices?

- $C_A > C_b$
- $C_A = C_b$
- $C_A < C_b$
So, in the solution of this very simple example the following elements of derivative pricing arise:

- **No-arbitrage assumption.** If there is another price in the market than the computed price, it is possible to earn money without taking risk.

- **Completeness.** If the model satisfies the no-arbitrage assumption and there exists a replication portfolio (or hedge strategy), then the price of the derivative is unique.

- **Equivalent Martingale Measure or Risk-Neutral Measure.** The probabilities did not appear to be of any interest to the solution of the problem. In fact, the solution could be rewritten such that the option price is an expectation under an equivalent measure. Under this measure, the discounted stock price is a martingale.

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In the Black-Scholes model this is all transferred to continuous time in the following way. Let \((\Omega, \mathbb{P}, \mathcal{F})\) be a probability space with filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) and \(W_t\) a standard Brownian motion w.r.t. this filtration.

- **Stock price process:** 
  \[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- **Risk-free process:** 
  \[ dB_t = r B_t dt \]

- **Trading (self-fin):** 
  \[ H_t = H_0 + \int_0^t \phi_u dS_u \]

- **Risk Neutral \(\mathbb{Q}\):** 
  \[ dS_t = r S_t dt + \sigma S_t d\tilde{W}_t \]
Option Pricing (4)

The price of an option with pay-off function \( \Phi(S_T) \) is given by:

\[
V_\Phi = e^{-rT} \mathbb{E}_Q \left[ \Phi(S_T) \right]
\]

For a call option this would be:

\[
V_{\text{call}} = e^{-rT} \mathbb{E}_Q \left[ (S_T - X)^+ \right]
\]

Replicating Portfolio :

\[
\Delta = \frac{dV_{\text{call}}(s_0)}{ds_0}
\]

Introduction and Motivation (1)

Suppose \( \{S_t\}_{t \geq 0} \) is a stock price process on some probability space \( \{\Omega, \mathcal{F}, \mathbb{P}\} \).

And that we are interested in a standard barrier option with Barrier \( L=90 \).
Suppose that we request the stock price process to \textit{consecutively} stay below the barrier for time $D=0.05$. Then, it is \textit{only} the green path that triggers the option at the \textbf{Parisian} stopping time.

We use the following notation for the Parisian stopping time, which represents

\textit{The first time the stock price process $S$ spends a consecutive time period of length $D$ below barrier $L$.}"

This represents a single-sided Parisian stopping time. We introduce the double-sided Parisian stopping time:

$$T_{D_1,D_2}^{L_1-,L_2+} := T_{D_1}^{L_1-} \wedge T_{D_2}^{L_2+}$$
Now we can construct a variety of Parisian options, e.g.

- **Double Parisian In Call**
  \[ V_{DPIC} = e^{-rT} \mathbb{E} \left[ \mathbb{1}_{\{T_{D_1,D_2}^{L_1,L_2+} \leq T\}} (S_T - K)^+ \right] \]

- **Double Parisian Out Put**
  \[ V_{DPOP} = e^{-rT} \mathbb{E} \left[ \mathbb{1}_{\{T_{D_1,D_2}^{L_1,L_2+} > T\}} (K - S_T)^+ \right] \]

- **Parisian Down and In Call**
  \[ V_{PDIC} = e^{-rT} \mathbb{E} \left[ \mathbb{1}_{\{T_{D_1,D_2}^{L_1,L_2+} \leq T\}} (S_T - K)^+ \right], \quad D_2 > T \]

**Introduction and Motivation (6)**

- Parisian Options are not exchange traded. Suggested to use instead of barrier options for illiquid stocks.

- Applications of Parisian Options:
  - Convertible Bonds (Screw clauses)
  - Default Risk of Life Insurance Companies
  - Real Options

- Double-Sided Parisian options:
  - Might have applications in some of the areas mentioned above
  - Are the most general type of Parisian options. Every contract can be derived from the Double-Sided Parisian In Call.
There is a lot of literature on the single-sided Parisian options. Here some pioneering work:

- **Laplace Transform**
  - (1997) Chesney, Jeanblanc and Yor, “Brownian Excursions and Parisian Barrier Options”

- **PDE approach**
  - (1999) Haber, Schonbucher and Wilmott, “Pricing Parisian Options”

- **Binomial Tree**

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**Theory (1)**

We assume the stock price process $S$ to be the standard GBM. Like Carr/Madan, we write the pricing formula:

$$ V_{DPIC} = e^{-rT} \mathbb{E} \left[ \mathbb{I}_{\left\{ T_{L_1,L_2} \leq T \right\}} (S_T - K)^+ \right] $$

In terms of probabilities:

$$ V_{DPIC} = S_0 P_{\tau + \sigma^2} (T) - K e^{-rT} P_{\tau} (T) $$

Where

$$ P_{\mu} (T) = \mathbb{P}_{\mu} \left[ S_T > K; T_{L_1,L_2} (S) \leq T \right], \quad \mu \in \mathbb{R} $$

Drift of GBM
Theory (2)

This probability is in terms of GBM. We translate it to a – due to Girsanov – “pseudo” probability in terms of standard Brownian motion:

\[ P_T(T) = e^{-\frac{1}{2}m^2T}E \left[ e^{mW_T}1\{W_T > k\}1\{\tau \leq T\} \right] \]

Where \( \tau \) is shorthand notation for the Parisian stopping time and \( m \) is the constant for the Girsanov transformation (zero for \( r = 1/2 \sigma^2 \))

There is no explicit formula for \( P_T(T) \), so we calculate its Fourier Transform:

\[ \phi(v) = \int_0^\infty e^{ivT}e^{-\alpha T}P_T(T)\,dT \]

Theory (3)

This Fourier Transform can be rewritten into,

\[ \phi(v) = \int e^{(iv-\alpha)T}E \left[ e^{mW_T}1\{W_T > k\}1\{\tau \leq T\} \left( 1\{\tau^+ < \tau^- \} + 1\{\tau^- < \tau^+ \} \right) \right] \,dT \]

\[ =: \phi_+(v) + \phi_-(v). \]

This, we can rewrite again as

\[ \phi_+(v) = E_+ (\tilde{\alpha})E \left[ \int_0^\infty e^{(iv-\alpha)\rho}h \left( \rho, l_2 + \sqrt{D_2}N \right) \,d\rho \right] \]

where

\[ E_+(\lambda) := E \left[ e^{-\frac{1}{2}\lambda^2\tau}1\{\tau^+ < \tau^- \} \right] \]
The key problem for the double-sided Parisian is to establish the independence of the pair \((W_t, \tau)\) and to calculate:

\[
E_+ (\lambda) := E \left[ e^{-\frac{1}{2} \lambda^2 \tau} 1_{\{\tau + < \tau^-\}} \right]
\]

The idea is to use the results of CJY on the single-sided Parisian option problem, combined with several application of the strong Markov property. Start with the Martingale:

\[
1 = E \left[ e^{-\frac{1}{2} \lambda^2 \tau + \lambda W_t} \right]
= E \left[ e^{-\frac{1}{2} \lambda^2 \tau + \lambda W_t} 1_{\{\tau + < \tau^-\}} \right] + E \left[ e^{-\frac{1}{2} \lambda^2 \tau + \lambda W_t} 1_{\{\tau^- < \tau\}} \right]
\]

We proceed by:

\[
E \left[ e^{-\frac{1}{2} \lambda^2 \tau + \lambda W_t} 1_{\{\tau + < \tau^-\}} \right] = E \left[ e^{-\frac{1}{2} \lambda^2 \tau + \lambda (\mu_t^2 n_t^2 + l_t^2)} 1_{\{\tau + < \tau^-\}} \right]
\]

\[
\mu_t^l = 1_{\{T_t < t\}} \text{sgn}(W_t - l) \sqrt{t - \gamma_t^l}
\]

\[
n_t^l = \frac{1_{\{T_t < t\}}}{\sqrt{t - \gamma_t^l}} |W_t - l|
\]
And by similar arguments we find:

$$
\mathbb{E}\left[ e^{-\frac{1}{2}\lambda^2 \tau + \lambda W_\tau} \mathbb{1}_{\tau^+ > \tau^-} \right] = e^{\lambda_1 \Psi(-\lambda \sqrt{D_1})} \mathbb{E}_-(\lambda)
$$

Substitution in the martingale equation gives:

$$
\mathbb{E}_+(\lambda) = \frac{e^{\lambda_1 \Psi(-\lambda_1)} - e^{-\lambda_1 \Psi(\lambda_1)}}{e^{\lambda_1 l_2 \Psi(-\lambda_1) - \Psi(\lambda_2)} - e^{\lambda_1 (l_2 - l_1) \Psi(\lambda_1) \Psi(\lambda_2)}}
$$

Where $\Psi$ is a special function related to the z-transform of $n_i$

A nice theoretical result is obtained by $\lambda$ to zero:

**Corollary 2.3** The probability that a Brownian motion will spend time $D_1$ below level $l_1$ before it spends $D_2$ above level $l_2$ is given by the following formula,

$$
P[\tau^- < \tau^+] = \frac{l_2 \sqrt{\frac{2}{\pi}} + \sqrt{D_2}}{(l_2 - l_1) \sqrt{\frac{2}{\pi}} + \sqrt{D_1} + \sqrt{D_2}} \quad l_1 < 0 < l_2.
$$

And now the barriers to zero gives the probability that a standard Brownian motion makes an excursion of length $D_2$ above zero before it makes an excursion below zero of length $D_1$.

$$
\frac{\sqrt{D_1}}{\sqrt{D_1} + \sqrt{D_2}}
$$
Practical Examples (1)

We can use the double-sided Parisian as a general option to compute the following Call contracts.

<table>
<thead>
<tr>
<th>Contract type</th>
<th>$S_0 = 90$</th>
<th>$S_0 = 95$</th>
<th>$S_0 = 100$</th>
<th>$S_0 = 105$</th>
<th>$S_0 = 110$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single P up-in (2a)</td>
<td>5.792</td>
<td>8.218</td>
<td>11.113</td>
<td>14.435</td>
<td>18.129</td>
</tr>
<tr>
<td>P up-before-down-in (3a)</td>
<td>3.568</td>
<td>6.844</td>
<td>10.284</td>
<td>13.957</td>
<td>17.886</td>
</tr>
<tr>
<td>Single P down-in (2b)</td>
<td>2.676</td>
<td>1.742</td>
<td>1.123</td>
<td>0.719</td>
<td>0.457</td>
</tr>
<tr>
<td>P down-before-up-in (3b)</td>
<td>2.668</td>
<td>1.723</td>
<td>1.087</td>
<td>0.651</td>
<td>0.339</td>
</tr>
</tbody>
</table>

Here we have $L_1 = 90$, $L_2 = 110$, $K = 100$, $\sigma = 25\%$, $r = 3.5\%$, $T=1$, $D_1=D_2=10/250$.

Practical Examples (2)

The following graph shows peculiar delta behavior.

Here we have $L_1 = 80$, $L_2 = 120$, $K = 100$, $\sigma = 25\%$, $r = 3.5\%$, $T=1$, $D_1=D_2=10/250$. 
Practical Examples (3)

Surprising behavior of theta:

<table>
<thead>
<tr>
<th></th>
<th>(S_0 = 70)</th>
<th>(S_0 = 74)</th>
<th>(S_0 = 76)</th>
<th>(S_0 = 80)</th>
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<tbody>
<tr>
<td>d</td>
<td>Par</td>
<td>Plain</td>
<td>Par</td>
<td>Plain</td>
</tr>
<tr>
<td>10</td>
<td>0.953</td>
<td>0.957</td>
<td>1.474</td>
<td>1.547</td>
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<td>9</td>
<td>0.946</td>
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<td>1.474</td>
<td>1.536</td>
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<td>8</td>
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<td>0.941</td>
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<tr>
<td>7</td>
<td>0.932</td>
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<td>6</td>
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<tr>
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<tr>
<td>1</td>
<td>0.885</td>
<td>0.885</td>
<td>1.448</td>
<td>1.448</td>
</tr>
</tbody>
</table>

Here we have \(L_1 = 80, L_2 = 120, K = 100, \sigma = 25\%, \ r = 3.5\%, \ T=1, \ D_1=D_2=10/250.\)

Final Remarks

This talk is only about a very tiny fraction of everything that is of interest in the field of financial mathematics.

In the field of financial mathematics many mathematical concepts, like equivalent measures, stochastic integrals fit one-to-one to the problem at hand.

There is an interplay between mathematical results and problems arising in the financial practice.

What is the impact to the field of the credit/financial crisis?